ELECTRONICS RESEARCH CENTER

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

PM-37

L_2-STABILITY OF TIME-VARYING SYSTEMS – CONSTRUCTION OF MULTIPLIERS WITH PRESCRIBED PHASE CHARACTERISTICS

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January 1968

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ABSTRACT

A system consisting of a linear element $G(s)$ and a time-varying gain $n(\cdot)$ is considered. It is assumed that this system is stable for all constant gains in the sector $[0,k]$ (i.e., for $n(t) \equiv \ell$, $0 \leq \ell \leq k$). It is then shown that the system is stable for all $n(\cdot)$ in that sector satisfying

$$\left| \frac{1}{t} \int_0^t \left| \frac{d}{d\tau} \left[\log \left(\frac{n(\tau)}{1 - \frac{n(\tau)}{k}} \right) \right] \pm 2\sigma \right| d\tau - 2\sigma \right| \leq \frac{K}{t}$$

for some $K > 0$ and all $t > 0$.

Here σ is a constant determined by an equation involving (inversely) the derivative of the phase function $\operatorname{Arg}\left\{G(i\omega) + 1/k\right\}$. The proof follows what are by now established lines in employing a "multiplier" operator. However a method is used to eliminate any "multiplier" dependence from the final results, so that these results are explicit and geometric.

1. INTRODUCTION

Many of the early "frequency domain" stability criteria afforded simple geometric interpretations. These criteria assured stability on the basis of the plot of a Nyquist curve and its position with respect to some other line or geometric figure. (Consider for instance the "Popov Criterion" or the "Circle Criterion".)

Some of the more recent work has resulted in theorems of a less geometric character. Brockett and Forys [1] considered a feedback system with time-invariant element G(s) and time-varying element $n(\cdot)$, $0 \le n(\cdot) \le k$ (see Figure 1) and concluded that if there exists some Z(s) of the form $Z(s) = \sum_{i=1}^{n} \alpha_i/(s+\beta_i)$ with $\alpha_i \ge 0$ and $\beta_i \ge \sigma$ and with $Z(s)^{\pm 1}[G(s) + 1/k]$ positive real, then the system would be stable for all $n(\cdot)$ satisfying $\dot{n}(t)/n(t) \le 2\sigma\,(1 - n(t)/k)$ for $t \ge 0$.

In the context of integral equations, a result of Falb and Zames [2] showed that for a system consisting of a convolution operator $\underset{\sim}{G}$ and a monotone non-linearity $f(\cdot)$ satisfying $0 \le \sigma f(\sigma) \le k\sigma^2$, stability can be proven if there exists an operator $\underset{\sim}{Z} : L_2[0,\infty) \to L_2[0,\infty)$ defined by $(\underset{\sim}{Z}x)(t) = x(t) + \int_0^t z_1(t-\tau)\,x(\tau)\,d\tau$ where

$$z_1(\cdot) \in L_1(0,\infty), \quad \int_0^\infty \left|z_1(t)\right| dt < 1$$

and

$$\mathrm{Re}\left\{[Z(i\omega) + \alpha i\omega]\ \left[G(i\omega) + \frac{1}{k}\right]\right\} \ge \delta > 0$$

for some $\alpha > 0$ and all real ω.

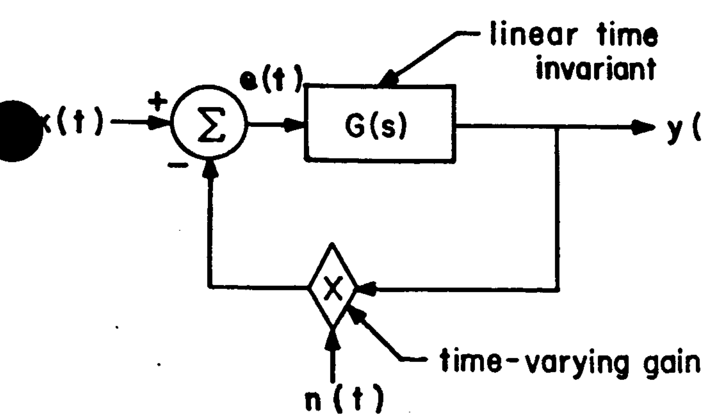

Figure 1. A Feedback System

Both of the preceding results depend on the existence of a "multiplier" operator which, when combined with [G(s) + 1/k], yields a positive operator. This idea, expressed as a factorization property of G(s), first appeared in Zames [3a]. The usefulness of the "multiplier" approach as portrayed in [1], [2], and [3a], is limited by the absence of any explicit method for finding suitable "multipliers".

In Freedman and Zames [4] the stability of a system involving a linear time-invariant element G(s) and a time-varying gain $n(\cdot)$ was considered (as in Figure 1). The method of proof in [4] involved introduction of a "multiplier" much as in [2] above. However, a constructive process was developed which allowed for removal of this "multiplier" from the final results, so that these results were geometric in nature.

It is this author's contention that [4] contains the core of a procedure which can be used to initiate a program aimed at returning more closely to the geometric character of the earlier results. More explicitly it is felt that the ideas in [4] can be utilized to remove the "multiplier" from many of the more recent stability results, thus yielding criteria depending only on G(s) and its properties.

For the purpose of this paper the result of Brockett and Forys [1] mentioned above was considered and a criterion was developed which is free from dependence on any "multiplier".

To describe this criterion more fully consider the feedback system represented by Figure 1, and assume it is stable for all constant gains in sector $[0, k]$. Denote by $\Phi(\omega)$ the phase function Arg $\left\{G(i\omega) + 1/k\right\}$. Then by the criterion to be presented here the system will be stable if there is a $K \geq 0$ such that

$$\left| \frac{1}{t} \int_o^t \left| \frac{\dot{n}(t)}{n(t)\,(1 - \frac{n(t)}{k})} \pm 2\sigma \right| dt - 2\sigma \right| \leq \frac{K}{t}$$

for all $t > 0$ (and so more weakly if $\dot{n}(t)/n(t) \leq 2\sigma\,(1 - n(t)/k)$ or $\dot{n}(t)/n(t) \geq 2\sigma\,(1 - n(t)/k)$) where σ is determined by an equation involving (see Figure 2):

1. The magnitude of the closest approach of $\Phi(\omega)$ to $\pm\,\pi$.
2. A "cutoff frequency W_ν" i.e. a frequency after which the values of $\Phi(\omega)$ are of no importance in this theory
3. The magnitude of the square integral of the derivative of $\Phi(\omega)$ over $[-W_\nu, W_\nu]$

One sees therefore that not only the geometric set $N = \left\{G(i\omega) \,\middle|\, -\infty < \omega < \infty\right\}$ comes into play here, but also in some sense the "angular" rate at which this set is traced out as ω varies (as represented by $\Phi'(\omega)$). In Fact it will appear that

$$\Phi(\omega) \triangleq \mathrm{Arg}\left\{G(i\omega) + \frac{1}{K}\right\}$$

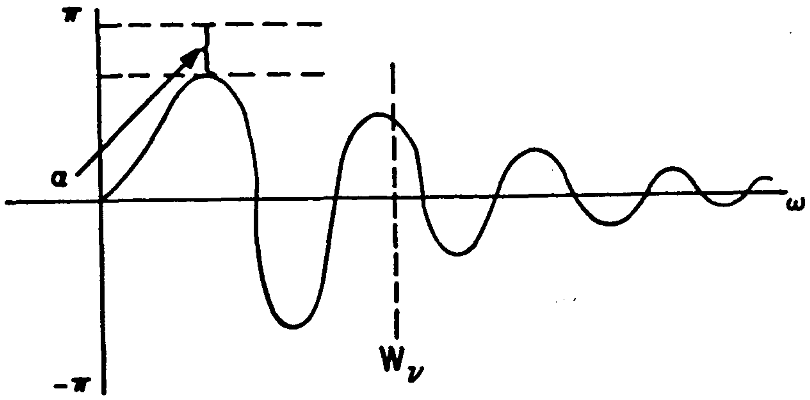

Figure 2. Plot of the Phase Function $\Phi(\omega)$

such rates are closely related to the existence and properties of certain "multipliers."

The crucial lemma on which the theory presented here rests is Lemma 1, section 4, which shows that a "multiplier" operator with any prescribed phase function can be constructed provided only that its phase function and its derivative satisfy suitable integrability conditions. This result appeared in [4] (Lemma 2, section 4), and the reader is referred to that paper for the proof. However, a short sketch of that proof is included here.

ACKNOWLEDGMENT

The author wishes to express his deep appreciation to Dr. G. Zames for introducing him to the areas considered here and for numerous suggestions and discussions concerning these topics.

2. THE MAIN PROBLEM AND ITS SOLUTION

DEFINITION: Let $L_p[0, \infty)$, where $p = 1, 2 \ldots \infty$, be the linear space of real valued measurable functions $x(\cdot)$ on $[0, \infty)$ with the property that

$$\int_0^\infty |x(t)|^p \, dt < \infty \text{ if } 1 \leq p < \infty,$$

or $x(\cdot)$ is essentially bounded if $p = \infty$. Let $L_2[0, \infty)$ be normed with the norm

$$\|x(\cdot)\| = \left\{ \int_0^\infty |x(t)|^2 \, dt \right\}^{1/2}$$

The spaces $L_p(-\infty, \infty)$ on the interval $(-\infty, \infty)$ are similarly defined.

The definition of the extended space L_{2e} is introduced next. (For a more complete discussion of such spaces, see [3b].)

DEFINITION: Let L_{2e} be the space of those real-valued measurable functions $x(\cdot)$ on $[0, \infty)$ satisfying

$$\int_0^T |x(t)|^2 \, dt < \infty \text{ for all } T \geq 0.$$

2.1 FEEDBACK EQUATIONS AND STABILITY

The feedback system of Figure 1 will be represented for all $t \geq 0$ by the integral equation

$$e(t) = x(t) - n(t)\left[\int_0^t e(\tau)\, g(t - \tau)\, d\tau\right] \tag{1}$$

or, alternatively the pair:

$$\left.\begin{aligned} e(t) &= x(t) - n(t)\, y(t) \\ y(t) &= \int_0^t e(\tau)\, g(t-\tau)\, d\tau \end{aligned}\right\} \qquad (2)$$

in which it is assumed that:

1. $x(\cdot)$ is in $L_2[0, \infty)$. (The function $x(\cdot)$ represents the combined effects of an input and of possible non-zero initial conditions.)
2. $g(\cdot)$ is in $L_1[0,\infty)$
3. $n(\cdot)$ is a real-valued function, absolutely continuous on $[0,\infty)$. (Since $n(\cdot)$ is absolutely continuous, its derivative $\dot{n}(\cdot)$ exists almost everywhere, and

$$n(b) - n(a) = \int_a^b \dot{n}(t)\, dt$$

for any non-negative real a and b (See Hobson [5], Sect. 406, pp. 592-593.)
4. $e(\cdot)$ (and also $y(\cdot)$) is in L_{2e} (i.e., existence of solutions in L_{2e} for $L_2[0,\infty)$ - inputs is being assumed.*

DEFINITION: Feedback system 1 will be termed L_2 - stable if for any pair $(x(\cdot), e(\cdot))$ for which (1) (or (2)) and the related assumptions 1, 2, 3 and 4 hold, then $e(\cdot)$ is in $L_2[0,\infty)$, with $||\ e(\cdot)|| \leq$ const. $||x(\cdot)||$.

*The problem of existence of L_{2e} solutions will not be discussed in this paper, except to say that such questions may be settled favorably by various minor additional hypotheses.

This notion of stability is natural in the setting of integral equations. In the context of differential equations it implies asymptotic stability ($\lim_{t \to \infty} y(t) = 0$) and with additional minor assumptions can also be used to show bounded-input, bounded-output stability.

2.2 THE MAIN STABILITY THEOREM

This section contains the main stability results. A few definitions and remarks will provide the setting.

DEFINITION: For any $k > 0$, let

$$W_k = \left\{ f(\cdot) \epsilon\ L_1[0,\infty) \middle|\ F(i\omega) + \frac{1}{\ell} \neq 0 \text{ for } -\infty < \omega < \infty \text{ and all } \ell,\ 0 < \ell < k \right\}$$

where $F(s)$, the Laplace transform of $f(\cdot)$, is the complex-valued function with domain $\left\{ s \middle| \mathrm{Re}\{s\} \geq 0 \right\}$ defined, as usual, by the integral

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt;$$

i.e.,

$f(\cdot) \epsilon W_k$ iff the set $\left\{ F(i\omega) \middle| \omega \epsilon (-\infty, \infty) \right\}$ does not cut the negative real axis from $-\infty$ up to and including the point $-\frac{1}{k}$.

REMARK 1: The statement "$g(\cdot) \epsilon W_k$" may be interpreted via the "principle of the argument" to be equivalent to the statement: "The equation $G(s) = -\frac{1}{k}$ has no complex roots in $\mathrm{Re}\{s\} \geq 0$." Also the classical Nyquist Criterion assures that "$g(\cdot)$ in W_k" is a necessary and sufficient condition for feedback system (1) to be L_2-stable for all constant gains between 0 and k; i.e., for $n(t) \equiv \ell$ where ℓ is a constant $0 \leq \ell \leq k$.

REMARK 2 AND SOME SPECIAL NOTATION: Given $g(\cdot) \epsilon W_k$, define the phase function

1. $\Phi(\omega) \triangleq \text{Arg}\left\{G(i\omega) + \frac{1}{k}\right\}$.

Then since $G(i\omega) + 1/k$ does not cut the negative real axis as ω passes from $-\infty$ to $+\infty$, it follows that $\Phi(\omega)$ is uniquely defined for all ω, and takes values only in $(-\pi, \pi)$. Further, since $g(\cdot) \epsilon L_1[0, \infty)$ the Riemann-Lebesgue Lemma assures that $\lim_{|\omega| \to \infty} G(i\omega) = 0$ and so $\lim_{|\omega| \to \infty} \text{Arg}\left\{G(i\omega) + 1/k\right\} = 0$ also. Therefore a simple continuity argument shows that the function $G(i\omega) + 1/k$ must have a "closest angular approach" α, to the negative real axis; that is, letting:

2. $\alpha \triangleq \min(\pi - |\Phi(\omega)|)$

then $\alpha > 0$.

Next let

3. $\nu = \frac{\pi - \alpha}{3}$

and define the "cutoff frequency"

4. $W_\nu \triangleq \min\left\{W / |\Phi(\omega)| \leq \nu \text{ for } |\omega| \geq W\right\}$.

Here the existence of W_ν is assured by an argument employing the continuity of $\Phi(\omega)$ and the fact that $\lim_{|\omega| \to \infty} \Phi(\omega) = 0$.

The importance of W_ν is that it represents a frequency value beyond which information about the phase function $\Phi(\omega)$ need not be utilized in the theory to follow.

In the remainder of this paper the notation introduced in (1) through (4) above will be used freely.

The main result may now be stated.

THEOREM 1: Suppose Eq. (1) (or equivalently Eqs. (2)) and the related assumptions (1 through 4) hold for a pair ($x(\cdot)$, $e(\cdot)$). Let $k > 0$ be given and assume:

1. $g(\cdot) \epsilon W_k$ (i.e., the system is L_2-stable for $n(t) \equiv \ell$ any ℓ, $0 \leq \ell \leq k$)

2. $0 < \inf n(t) \leq n(t) \leq \sup n(t) < k$. Then:

   (a) if $|\Phi(\omega)| \leq \pi/2$ for $-\infty < \omega < \infty$, the system is L_2-stable by a simple possivity argument. (Note that $|\Phi(\omega)| \leq \pi/2$ for $-\infty < \omega < \infty$ implies $\mathrm{Re}\{G(i\omega)\} \geq -1/k$ for $-\infty < \omega < \infty$ and the desired result follows from the basic Popov theorem).

On the other hand:

   (b) If $\Phi(\omega) > \pi/2$ for some ω, the "cutoff frequency" W_ν is strictly positive and one may define

$$\sigma_* \triangleq \frac{\left(\frac{3\pi\alpha}{16}\right)^2}{\int_{-W_\nu}^{+W_\nu} |\Phi'(\omega)|^2 \, d\omega}$$

where $\Phi'(\omega)$ denotes the derivative of the phase function $\Phi(\omega)$.

Suppose then that there exists a constant $K > 0$ and a constant σ in $(0, \sigma_*)$ such that

$$\left| \frac{1}{t} \int_0^t \left| \frac{\dot{n}(t)}{n(t)\left(1 - \frac{n(t)}{k}\right)} - 2\sigma \right| dt - 2\sigma \right| \leq \frac{K}{t} \qquad (3)$$

for all $t > 0$ or else if

$$\left| \frac{1}{t} \int_o^t \left| \frac{\dot{n}(t)}{n(t)\left(1 - \frac{n(t)}{k}\right)} + 2\sigma \right| dt - 2\sigma \right| \leq \frac{K}{t} \tag{4}$$

for all $t > 0$. Then $e(\cdot)$ is in $L_2[0,\infty)$ and, in fact, $\| e(\cdot) \| \leq$ const. $\| x(\cdot) \|$. Therefore System (1) is L_2-stable.

REMARK 3: Theorem 1 is an immediate consequence of Theorem 2, Section 3, Lemma 3 and Remark 4, Section 4, and Corollary 3, Section 5.

COROLLARY 1: Under the assumptions and notation of Theorem 1 a sufficient condition on $n(\cdot)$ for (3) to hold, and hence for L_2-stability, is that

$$\frac{\dot{n}(t)}{n(t)} \leq 2\sigma\left(1 - \frac{n(t)}{k}\right).$$

for all $t > 0$. Similarly the inequality

$$\frac{\dot{n}(t)}{n(t)} \geq -2\sigma\left(1 - \frac{n(t)}{k}\right)$$

is a sufficient condition for (4) to hold and so also to ensure L_2-stability. (See R. W. Brockett and L. J. Forys [1] for a "multiplier" result along these lines.)

PROOF OF COROLLARY 1: Immediate.

COROLLARY 2: Let the hypotheses and notation of Theorem 1 remain valid except in Case (b) redefine σ_* as follows:

$$\sigma_* \triangleq \left(\frac{3\pi\alpha}{16\sqrt{2}}\right)^2 \cdot \frac{1}{\max\limits_{-W_\nu \leq \omega \leq W_\nu} |\Phi'(\omega)|^2}$$

Then with this definition of σ_* the conclusions of the Theorem 1 and also of Corollary 1 remain valid; i.e., System (1) is L_2-stable.

PROOF OF COROLLARY 2: This choice of σ_* is less than or equal to the σ_* of the original statement.

3. A THEOREM ON MULTIPLIERS

The following theorem is essentially proved in Zames-Freedman [4] and the reader is recommended to that paper for the proof. The existence of a multiplier operator $\underset{\sim}{Z}$ satisfying certain properties with respect to $\underset{\sim}{G} + \frac{1}{k}$ is hypothesized and a stability conclusion is then drawn for all time-varying gains $n(\cdot)$ suitably restricted. The result in the form presented below is rather similar to a result in Brockett-Forys [1]; however it differs in one sense in that the form of the multiplier $\underset{\sim}{Z}$ remains unspecified in this paper. Moreover, the key fact to keep in mind here is that this theorem represents an intermediate stage in the developments leading to Theorem 1 stated in the previous section, and Theorem 1 contains no explicit mention of the multiplier.

For what follows the definition of some special operator spaces (actually they are Banach Algebras) will be of use.

DEFINITION: Let $\mathcal{L}_{\sigma_o}$ be the class of operators. $\underset{\sim}{H}$: $L_{2e} \to L_{2e}$ satisfying

$$(\underset{\sim}{H}x)(t) = h_o\, x(t) + \int_o^t x(\tau) h_1(t - \tau)\, d\tau$$

(for all $x(\cdot) \epsilon L_{2e} [0, \infty)$ and all $t \geq 0$) where h_o is a real constant and $h_1(\cdot)$ is a real-valued measurable function on $0, \infty)$ satisfying $h_1(t) \exp (\sigma_o t) \epsilon L_1[0, \infty)$.

For $\underset{\sim}{H} \epsilon \mathcal{L}_{\sigma_o}$ the Laplace transform of $\underset{\sim}{H}$ is given by

$$H(s) = h_o + \int_o^\infty h_1(t) \exp(-st)\, dt$$

(for all complex s with Re $\{s\} \geq -\sigma_o$).

THEOREM 2: Let Eq. (1) (or equivalently Eqs. (2)) and the related assumptions (1 through 4) hold for a pair $(x(\cdot), e(\cdot))$. Let $k > 0$ be given and assume

$$0 < \inf n(t) \leq n(t) \leq \sup n(t) < k.$$

Further, let the following two assumptions be made:

1. There is a constant $\sigma > 0$ and an operator $\underset{\sim}{Z} \in \mathscr{L}_\sigma$ satisfying, for all real ω,

   (i) $\operatorname{Re}\left\{Z(i\omega - \sigma)\right\} \geq 0$

   (ii) $\operatorname{Re}\left\{Z(i\omega)\left[G(i\omega) + \frac{1}{k}\right]\right\} \geq \delta$

   for some positive constant δ.

2. The function $n^*(\cdot)$ defined by $n^*(t) \triangleq n(t)(1 - \frac{n(t)}{k})^{-1}$ for all $t \geq 0$, or the reciprocal of this function, may be factored into a product $n_1(\cdot) \cdot n_2(\cdot)$ of two absolutely continuous functions on $[0, \infty)$; where $n_1(t) \exp(-2\sigma t)$ is monotone non-increasing while $n_2(t)$ is monotone non-decreasing in t, and $0 < \inf n_1(t) \leq \sup n_2(t) < \infty$ (and so also $0 < \inf n_2(t) \leq \sup n_2(t) < \infty$).

Then the system (1) is L_2-stable, i.e.,

$$e(\cdot) \text{ is in } L_2[0, \infty) \text{ and } \|e(\cdot)\| \leq \text{const. } \|x(\cdot)\|$$

PROOF OF THEOREM 2: See Lemma 1, Zame-Freedman [4]. The factorization property (imposed on the time-varying gain) hypothesized in [4] is slightly different from the one made in this present paper, but the proof under the conditions considered here follows the same general arguments and is, in fact, a bit easier. For that reason the reader is referred to [4].

4. MULTIPLIERS WITH PRESCRIBED PHASE CHARACTERISTICS

In this section three lemmas will be presented which together settle the question of the existence of a suitable multiplier for Theorem 2. Such a multiplier will be seen to always exist and, in fact, an associated range of σ values (as required for the hypothesis of Theorem 2) may be obtained directly from data concerning the phase of $G(i\omega) + \frac{1}{k}$ without recourse to construction of $\underset{\sim}{Z}$ in any given case.

The foundation of the results to follow in this section is Lemma 1 below, which assures the existence of an operator $\underset{\sim}{Z}$ in $\mathcal{L}_\sigma$ with any prescribed phase function $\Phi_o(\omega) = \text{Arg}\left\{Z(i\omega)\right\}$ provided only the function $\Phi_o(\omega)$ and its derivative $\Phi_o'(\omega)$ satisfy certain integrability conditions.

LEMMA 1: (OPERATORS WITH PRESCRIBED PHASE)

If:

1. $\Phi_o(\omega)$ is a real-valued continuous a. e. differentiable odd function of ω for ω in $(-\infty, \infty)$
2. $\Phi_o(\omega)$ and $\Phi_o'(\omega)$ are in $L_2(-\infty, \infty)$,

then

(a) there is a function $\lambda(\cdot)$ in $L_1(-\infty, \infty)$ with $\lambda(t) = 0$ for $t < 0$ and with a Laplace transform $\Lambda(s)$ satisfying $\text{Im}\left\{\Lambda(i\omega)\right\} = \Phi_o(\omega)$.

(b) there is a $y(\cdot)$ in $L_1(-\infty, \infty)$ with $y(t) = 0$ for $t < 0$, and with a Laplace transform $Y(s)$ satisfying $1 + Y(s) = \exp[\Lambda(s)]$ for $\text{Re}\left\{s\right\} \geq 0$.

(c) If $-\pi < \Phi_o(\omega) < \pi$, there is a $y(\cdot) \epsilon L_1(-\infty, \infty)$ with $y(t) = 0$ for $t < 0$, $1 + Y(s) \neq 0$ in $\text{Re}\left\{s\right\} \geq 0$ (so $1 + Y(s)$ is minimum phase) and $\text{Arg}\left\{1 + Y(i\omega)\right\} = \Phi_o(\omega)$.

OUTLINE OF PROOF: For the complete proof of this lemma the reader is once again referred to Zames-Freedman [4]. However a brief outline of the main ideas is presented here.

(i) Let $\phi_o(t)$ denote the inverse limit-in-the-mean Fourier transform of $\Phi_o(\omega)$ and define

$$\lambda(t) = \begin{cases} 2\phi_o(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Then it may be deduced from 1 and 2 that $\lambda(\cdot)$ is in $L_1(-\infty, \infty)$ and that the Laplace transform of $\lambda(\cdot)$ denoted $\Lambda(s)$ satisfies $\mathrm{Im}\left\{\Lambda(i\omega)\right\} = \Phi_o(\omega)$ and so (a) above holds.

(ii) Next, for each $n \geq 0$, let

$$y_n(t) = \lambda(t) + \frac{(\lambda * \lambda)(t)}{2!} + \ldots \frac{\overbrace{(\lambda * \ldots * \lambda)}^{n}(t)}{n!}$$

where $*$ represents convolution; that is:

$$(x_1 * x_2)(t) \triangleq \int_{-\infty}^{\infty} x_1(t - \tau)\, x_2(\tau)\, d\tau.$$

Then, for each n, $y_n(\cdot)$ is in $L_1[0, \infty)$ and the Laplace transform of $y_n(\cdot)$, $Y_n(s)$ satisfies

$$Y_n(s) = \Lambda(s) + \frac{\Lambda^2(s)}{2!} + \ldots \frac{\Lambda^n(s)}{n!}$$

(iii) Finally it may be shown that $y_n(\cdot)$ converges in L_1-norm to a function $y(\cdot)$ in $L_1[0, \infty)$ and the Laplace transform of this function $y(\cdot)$, denoted $Y(s)$, must satisfy

$$Y(s) = \exp\left[\Lambda(s)\right] - 1 \text{ for } \mathrm{Re}\left\{s\right\} \geq 0.$$

From this fact (b) and (c) of Lemma 1 follow.

In applying Lemma 1 to the problem of constructing a multiplier $\underset{\sim}{Z}$ which will meet the hypotheses of Theorem 2 it is clear that the larger σ may be chosen (with a corresponding $\underset{\sim}{Z} \in \mathcal{L}_\sigma$ assured), the less restrictive will be the conditions on $n(\cdot)$. A measure of how large a σ is possible will be developed via the following lemma. This lemma in its face concerns the rate of convergence of the values of the harmonic function Im $\{\Lambda(s)\}$ defined on a half plane Re $\{s\} \geq 0$ as s approaches Re $\{s\} = 0$ along ordinate lines.

LEMMA 2: Under the same notation and assumptions as in the previous theorem, it follows that for any $\sigma > 0$.

$$\sup_{-\infty < \omega < \infty} \left| \operatorname{Im}\left\{\Lambda(i\omega + \sigma)\right\} - \Phi_o(\omega) \right| \leq \frac{16\sqrt{\sigma}}{3\pi} \left[\int_{-\infty}^{\infty} \left| \Phi_o'(\omega) \right|^2 d\omega \right]^{\frac{1}{2}}$$

and so if $-\pi < \Phi_o(\omega) < \pi$ for all ω and if the principal value of Arg $\{\quad\}$ is taken, then:

$$\sup_{-\infty < \quad < \infty} \left| \operatorname{Arg}\left\{1 + Y(i\omega + \sigma)\right\} - \Phi_o(\omega) \right| \leq \frac{16\sqrt{\sigma}}{3\pi} \left[\int_{-\infty}^{\infty} \left| \Phi_o'(\omega) \right|^2 d\omega \right]^{\frac{1}{2}}$$

PROOF OF LEMMA 2: (The proof presented here follows closely the lines of the singular integral theory of Titchmarsh [6], Sect 116, pp 28-29). For all s with Re $\{s\} \geq 0$ the Laplace transform

$$\Lambda(s) \triangleq \int_0^{\infty} \exp(-st)\, \lambda(t)\, dt \tag{5}$$

Now let, for the moment,

$$q(t) = \begin{cases} \exp(-st) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

So, for any s with $\mathrm{Re}\{s\} > 0$ the Fourier transform of $q(\cdot)$ is $\frac{1}{i\omega + s}$. Applying Parseval's Theorem to (5) results in

$$\Lambda(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\omega + s}\, \overline{\Lambda(i\omega)}\, d\omega \tag{6}$$

However, applying Cauchy's Theorem to the analytic function $\Lambda(s)$ yields

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\omega + s}\, \Lambda(i\omega)\, d\omega \tag{7}$$

for $\mathrm{Re}\{s\} > 0$.

Now $\Phi_o(\omega) = \mathrm{Im}\{\Lambda(i\omega)\} = \frac{\Lambda(i\omega) - \overline{\Lambda(i\omega)}}{2i}$ as seen in the previous lemma and so subtracting (6) from (7) gives:

$$\Lambda(s) = -\frac{i}{\pi}\int_{-\infty}^{\infty} \Phi_o(\omega)\, \frac{1}{s + i\omega}\, d\omega \tag{8}$$

for $\mathrm{Re}\{s\} > 0$.

Next, for convenience of notation, letting $s = \sigma + iy$, $\bar{s} = \sigma - iy$ and $\mathrm{Im}\{\Lambda(s)\} = V(\sigma, y)$, one obtains:

$$V(\sigma, y) = \frac{1}{2i}\left[-\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\Phi_o(\omega)}{s + i\omega} + \frac{\Phi_o(\omega)}{\bar{s} - i\omega}\right] d\omega$$

for $\mathrm{Re}\{s\} > 0$

and so

$$V(\sigma, y) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma}{\sigma^2 + (y + \omega)^2} \Phi_o(\omega)\, d\omega$$

for $\sigma > 0$

or replacing ω by $-\omega$ and noting that $\Phi_o(\omega) = -\Phi_o(-\omega)$ gives:

$$V(\sigma, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma}{\sigma^2 + (y - \omega)^2} \Phi_o(\omega)\, d\omega \tag{9}$$

for $\sigma > 0$

Next note that for any $y \in (-\infty, \infty)$

$$\frac{1}{\pi} \int_{-y}^{\infty} \frac{\sigma}{\sigma^2 + (y - \omega)^2}\, d\omega = \frac{1}{\pi} \int_{-\infty}^{y} \frac{\sigma}{\sigma^2 + (y - \omega)^2}\, d\omega = \frac{1}{2}.$$

Therefore:

$$\left| \frac{1}{\pi} \int_{y}^{\infty} \frac{\sigma}{\sigma^2 + (y - \omega)^2} \Phi_o(\omega)\, d\omega - \frac{\Phi_o(y)}{2} \right.$$

$$= \left| \frac{1}{\pi} \int_{y}^{\infty} \frac{\sigma}{\sigma^2 + (y - \omega)^2} \left[\Phi_o(\omega) - \Phi_o(y) \right] d\omega \right| \tag{10}$$

Now for any $\sigma > 0$ and any ω and y with $\omega \neq y$:

$$\frac{\sigma}{\sigma^2 + (y - \omega)^2} \leq \frac{1}{\sigma} \quad \text{and} \quad \frac{\sigma}{\sigma^2 + (y - \omega)^2} \leq \frac{\sigma}{(y - \omega)^2}$$

Using these relations to bound (10) results in

$$(10) \leq \frac{1}{\pi\sigma}\int_y^{y+\sigma} \left| \Phi_o(\omega) - \Phi_o(y) \right| d\omega + \frac{\sigma}{\pi}\int_{-y+\sigma}^{\infty} \frac{\left|\Phi_o(\omega) - \Phi_o(y)\right|}{(y-\omega)^2}\, d\omega \qquad (11)$$

The first integral on the right hand side of (11) satisfies the following chain of inequalities:

$$\frac{1}{\pi\sigma}\int_y^{y+\sigma} \left|\Phi_o(\omega) - \Phi_o(y)\right| d\omega \leq \frac{1}{\pi\sigma}\int_y^{y+\sigma}\int_y^{\omega} \left|\Phi_o'(t)\right| dt\, d\omega$$

$$\leq \frac{1}{\pi\sigma}\int_y^{y+\sigma}\left(\left[\int_y^{\omega}\left|\Phi_o'(t)\right|^2 dt\right]^{\frac{1}{2}} (\omega - y)^{\frac{1}{2}}\right) d\omega \leq \left(\int_{-\infty}^{\infty}\left|\Phi_o'(t)\right|^2 dt\right)^{\frac{1}{2}} \frac{1}{\pi\sigma}\cdot\frac{2\sigma^{3/2}}{3}$$

$$= \frac{2}{3\pi}\sqrt{\sigma}\left(\int_{-\infty}^{\infty}\left|\Phi_o'(t)\right|^2 dt\right)^{\frac{1}{2}} \qquad (12)$$

The second integral on the right hand side of (11) also satisfies a chain of inequalities, as follows:

$$\frac{\sigma}{\pi}\int_{y+\sigma}^{\infty} \frac{\left|\Phi_o(\omega) - \Phi_o(y)\right|}{(\omega - y)^2}\, d\omega = \frac{\sigma}{\pi}\int_{\sigma}^{\infty}\frac{\left|\Phi_o(y+t) - \Phi_o(y)\right|}{t^2}\, dt$$

$$\leq \frac{\sigma}{\pi}\int_{\sigma}^{\infty}\frac{1}{t^2}\left[\int_y^{y+t}\left|\Phi_o'(\xi)\right| d\xi\right] dt$$

$$\leq \left(\int_{-\infty}^{\infty}\left|\Phi_o'(\xi)\right|^2 d\xi\right)^{1/2}\cdot\frac{\sigma}{\pi}\cdot\int_{\sigma}^{\infty} t^{-3/2}\, dt$$

$$\leq \left(\int_{-\infty}^{\infty} \left| \Phi_o'(\xi) \right|^2 d\xi \right)^{1/2} \cdot \frac{\sigma}{\pi} \cdot \frac{2}{\sqrt{\sigma}}$$

$$= \left(\int_{\infty}^{\infty} \left| \Phi_o'(\xi) \right|^2 d\xi \right)^{1/2} \cdot \frac{2\sqrt{\sigma}}{\pi} \qquad (13)$$

Combining (11), (12) and (13) with (10) one finds that

$$\left| \frac{1}{\pi} \int_{y}^{\infty} \frac{\sigma}{\sigma^2 + (y - \omega)^2} \Phi_o(\omega)\, d\omega - \frac{\Phi_o(y)}{2} \right|$$

$$\leq \frac{8\sqrt{\sigma}}{3\pi} \left(\int_{-\infty}^{\infty} \left| \Phi_o'(\xi) \right|^2 d\xi \right)^{1/2} \qquad (14)$$

Now by a completely analogous procedure, one also has

$$\left| \frac{1}{\pi} \int_{-\infty}^{y} \frac{\sigma}{\sigma^2 + (y - \omega)^2} \Phi_o(\omega)\, dw - \frac{\Phi_o(y)}{2} \right|$$

$$\leq \frac{8\sqrt{\sigma}}{3\pi} \left(\int_{-\infty}^{\infty} \left| \Phi_o'(\xi) \right|^2 d\xi \right)^{1/2} \qquad (15)$$

Finally (14) and (15) together with (9) yield:

$$\sup_{-\infty < y < \infty} \left| V(\sigma, y) - \Phi_o(y) \right| \leq \frac{16\sqrt{\sigma}}{3\pi} \left(\int_{-\infty}^{\infty} \left| \Phi_o'(\xi) \right|^2 d\xi \right)^{1/2}$$

thus completing the proof.

The following lemma completes the "constructive" process for the $\underset{\sim}{Z}$ hypothesized in Theorem 2 and also gives a range of applicable σ values.

LEMMA 3: Let the hypotheses and notation of Theorem 1 covering g(·) hold; i.e., $g(\cdot)\in W_k$, $\Phi(\omega) \triangleq \operatorname{Arg}\left\{G(i\omega)+\frac{1}{k}\right\}$, $\alpha \triangleq \min\left(\pi - \left|\Phi(\omega)\right|\right) > 0$, $\nu \triangleq \pi - \alpha/3$ and $W_\nu = \inf\left\{W_\nu \;\middle|\; \left|\Phi(\omega)\right| \le \nu \text{ for } \left|\omega\right| \ge W\right\}$ Assume $W_\nu > 0$ and let

$$\sigma_* \triangleq \left(\frac{3\pi\alpha}{16}\right)^2 \cdot \frac{1}{\left[\int_{-W_\nu}^{W_\nu} \left|\Phi'(\omega)\right|^2 d\omega\right]^{1/2}}$$

Then for any σ in $[0, \sigma^*)$ there is a $y(\cdot)$ in $L_1[0,\infty)$, with Laplace transforms $Y(s)$, and a $\delta > 0$, such that

(i) $\operatorname{Re}\left\{1 + Y(i\omega)\right\} \ge \delta > 0$

(ii) $\operatorname{Re}\left\{[1 + Y(i\omega + \sigma)]\left[G(i\omega) + \frac{1}{k}\right]\right\} \ge \delta > 0.$

REMARK 4: If $\underset{\sim}{Z} \in \mathcal{L}_\sigma$ is defined by

$$(\underset{\sim}{Z}x)(t) = x(t) + \int_0^t y(t-\tau)\, e^{-\sigma(t-\tau)} x(\tau)\, d\tau,$$

then the following conditions are equivalent to (i) and (ii) above.

(i') $\operatorname{Re}\left\{Z(i\omega - \sigma)\right\} \ge \delta > 0$

(ii') $\operatorname{Re}\left\{Z(i\omega)\cdot\left[G(i\omega) + \frac{1}{k}\right]\right\} \ge \delta > 0.$

Hence condition I of Theorem 2 is satisfied for all σ in $(0, \sigma_*)$.

The hypothesis $g(\cdot) \epsilon W_k$ assures that $\text{Arg}\left\{G(i\omega) + \frac{1}{k}\right\}$ lies in the interval $(-\pi, \pi)$. If conclusions (i) and (ii) above are to be fulfilled, then it is clearly necessary that $\text{Arg}\left\{1 + Y(i\omega)\right\}$ lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and also $\text{Arg}\left\{\left[1 + Y(i\omega+\sigma)\right]\left[G(i\omega) + 1/k\right]\right\}$ lies in $(-\pi/2, \pi/2)$. That these two conditions are also sufficient for the validity of (i) and (ii) can be seen from the following.

REMARK 5: If $F(\omega)$ is a continuous complex-valued function on $(-\infty, \infty)$, $F(\omega) \neq 0$, $\left|\text{Arg}\left\{F(\omega)\right\}\right| < \pi/2$ and $\lim_{|\omega| \to \infty} F(\omega)$ exists and is a real contant greater than zero, then there is a constant $\delta > 0$ with the property that $\text{Re}\left\{F(\omega)\right\} \geq \delta$.

Recalling that $\text{Arg}\left\{[1 + Y(i\omega+\sigma)]\left[G(i\omega) + \frac{1}{k}\right]\right\} = \text{Arg}\left\{1 + Y(i\omega+\sigma)\right\} + \text{Arg}\left\{G(i\omega) + 1/k\right\}$, an initial attempt at constructing $y(\cdot)$ might be to employ Lemma 1 in order to find a $y(\cdot) \epsilon L_1(0,\infty)$ with $\text{Arg}\left\{1 + Y(i\omega)\right\} = -\frac{1}{2}\text{Arg}\left\{G(i\omega) + \frac{1}{k}\right\}$. With such a choice for $y(\cdot)$, both $\text{Arg}\left\{1 + Y(i\omega)\right\}$ and $\text{Arg}\left\{1 + Y(i\omega)\right\} + \text{Arg}\left\{G(i\omega) + 1/k\right\}$ $\left(= -1/2\ \text{Arg}\left\{G(i\omega) + 1/k\right\}\right)$ would lie in $(-\pi/2, \pi/2)$. The construction which will be adopted here must differ from the choice of phase function suggested above for large ω in order to meet the conditions required for application of Lemmas 1 and 2; namely, that the phase function chosen and its derivative must both have finite square integrals.

PROOF OF LEMMA 3: For each $\epsilon > 0$ choose $\ell_\epsilon(\omega)$ a continuous, a.e., differentiable real valued function defined on (W_ν, ∞) and satisfying:

(a) $\ell_\epsilon(W_\nu) = -\frac{\Phi(W_\nu)}{2} = -\frac{\text{Arg}\left\{G(i\omega) + \frac{1}{k}\right\}}{2}$

(b) $\left|\ell_\epsilon(\omega)\right| \leq \frac{\nu}{2}$ for all $\omega \epsilon (W_\nu, \infty)$

(c) $\int_{W_\nu}^{\infty} \left|\ell_\epsilon(\omega)\right|^2 d\omega < \infty$, and

(d) $\int_{W_\nu}^{\infty} \left|\ell_\epsilon'(\omega)\right|^2 d\omega < \frac{\epsilon^2}{2}$.

(Such functions are easily constructable. In fact, $\ell_\epsilon(\omega)$ may be chosen to be linear from

$$\omega = W_\nu \quad \text{to} \quad \omega_o(\epsilon) = \frac{3W_\nu^2 + W_\nu}{\epsilon^2}$$

with

$$\ell_\epsilon(\omega) = 0 \text{ for } \omega \geq \omega_o(\epsilon).)$$

For any $\epsilon > 0$, define

$$\Phi_\epsilon(\omega) = \begin{cases} -\frac{\Phi(\omega)}{2} = -\frac{1}{2}\text{Arg}\left\{G(i\omega) + \frac{1}{k}\right\} & \text{for } \omega \text{ in } [-W_\nu, W_\nu] \\ \ell_\epsilon(\omega) & \text{for } \omega > W_\nu \\ -\ell_\epsilon(-\omega) & \text{for } \omega < W_\nu \end{cases}$$

By this construction it is assured that

$$\left|\Phi_\epsilon(\omega)\right| < \frac{\pi}{2} \tag{16}$$

and

$$\left|\Phi_\epsilon(\omega) + \operatorname{Arg}\left\{G(i\omega) + \frac{1}{k}\right\}\right| < \frac{\pi}{2}. \tag{17}$$

Also for future reference the inequality

$$\int_{-\infty}^{\infty} \left|\Phi_\epsilon'(\omega)\right|^2 d\omega \leq \frac{1}{4} \int_{-W_\nu}^{+W_\nu} \left|\Phi'(\omega)\right|^2 d\omega + \epsilon^2 \tag{18}$$

holds.

It now follows by application of Lemma 1 that there is a $y_\epsilon(\cdot)$ in $L_1(0,\infty)$ with Laplace transform $Y(s)$, $1 + Y_\epsilon(i\omega) \neq 0$ any ω and $\operatorname{Arg}\left\{1 + Y_\epsilon(i\omega)\right\} = \Phi_\epsilon(\omega)$. Remark 5 combined with (16) therefore yields:

$$\operatorname{Re}\left\{1 + Y_\epsilon(i\omega)\right\} \geq \delta(\epsilon) > 0$$

for some constant $\delta(\epsilon)$. This proves (i). In fact (i) holds with $y(\cdot)$ equal to any $y_\epsilon(\cdot)$ chosen by the above procedure. The actual choice of $y_\epsilon(\cdot)$ which will be made will be one corresponding to ϵ sufficiently small so as to assure that (ii) also holds.

In order for (ii) to hold for $y(\cdot)$ equal to some $y_\epsilon(\cdot)$, it is sufficient, by Remark 5, to show that

$$\left|\operatorname{Arg}\left\{1 + Y_\epsilon(i\omega + \sigma)\right\} + \Phi(\omega)\right| < \frac{\pi}{2} \qquad \text{for } -\infty < \omega < \infty \tag{19}$$

Now (19) may be rewritten as

$$\left|\operatorname{Arg}\left\{1 + Y_\epsilon(i\omega + \sigma)\right\} - \Phi_\epsilon(\omega) + \Phi_\epsilon(\omega) + \Phi(\omega)\right| < \frac{\pi}{2}$$

$$\text{for } -\infty < \omega < \infty \tag{20}$$

From Lemma 2 it follows that for all ω

$$\left|\operatorname{Arg}\left\{1 + Y_\epsilon(i\omega + \sigma)\right\} - \Phi_\epsilon(\omega)\right| \le \frac{16\sqrt{\sigma}}{3\pi}\left[\int_{-\infty}^{\infty}\left|\Phi'_\epsilon(\omega)\right|^2 d\omega\right]^{1/2} \tag{21}$$

and so a sufficient condition for (20) to hold is that

$$\frac{16\sqrt{\sigma}}{3\pi}\left[\int_{-\infty}^{\infty}\left|\Phi'_\epsilon(\omega)\right|^2 d\omega\right]^{1/2} + \sup_{-\infty<\omega<\omega}\left|\Phi'_\epsilon(\omega) + \Phi(\omega)\right| < \frac{\pi}{2} \tag{22}$$

Now

$$\sup_{\omega,\ |\omega| \le W_\nu}\left|\Phi_\epsilon(\omega) + \Phi(\omega)\right| = \sup_{\omega,\ |\omega| \le W_\nu}\frac{|\Phi(\omega)|}{2} \le \frac{\pi - \alpha}{2}$$

while

$$\sup_{\omega,\ |\omega| \ge W_\nu}\left|\Phi_\epsilon(\omega) - \Phi(\omega)\right| \le \sup_{\omega,\ |\omega| \ge W_\nu}\left|\Phi_\epsilon(\omega)\right| + \sup_{\omega,\ |\omega| \ge W_\nu}\left|\Phi(\omega)\right|$$

$$\le \frac{\nu}{2} + \nu = \frac{3\nu}{2} = \frac{\pi - \alpha}{2}$$

It follows that

$$\sup_{-\infty<\omega<\infty}\left|\Phi_\epsilon(\omega) - \Phi(\omega)\right| \le \frac{\pi - \alpha}{2}\ .$$

Using this relation in conjunction with inequalities (20) and (21), one finds that a sufficient condition for (19) to hold is

$$\frac{16\sqrt{\sigma}}{3\pi}\left[\int_{-\infty}^{\infty}\left|\Phi_{\epsilon}'(\omega)\right|^{2} d\omega\right]^{1/2} + \frac{\pi-\alpha}{2} < \frac{\pi}{2}$$

or

$$\frac{16\sqrt{\sigma}}{3\pi}\left[\int_{-\infty}^{\infty}\left|\Phi_{\epsilon}'(\omega)\right|^{2} d\omega\right]^{1/2} < \frac{\alpha}{2} \tag{23}$$

Therefore if σ_* is chosen equal to

$$\frac{\left(\frac{3\pi\alpha}{16}\right)^2}{\int_{-W_\nu}^{W_\nu}\left|\Phi'(\omega)\right|^2 d\omega}$$

as hypothesized in this lemma, then for any σ, $0 < \sigma < \sigma_*$ the inequality

$$\left[\int_{-W_\nu}^{W_\nu}\left|\Phi'(\omega)\right|^2 d\omega\right]^{1/2} \cdot \frac{16\sqrt{\sigma}}{3\pi} < \alpha$$

holds.

But then if ϵ is chosen sufficiently small, inequality (23) will hold by virtue of (18). This means, on tracing through the above argument, that the $y_\epsilon(\cdot)$ corresponding to such a choice of ϵ will be an acceptable choice for the $y(\cdot)$ in the statement of this lemma; i.e., (i) and (ii) of the statement of this lemma hold for $Y_\epsilon(s)$, the Laplace transform of $y_\epsilon(\cdot)$. This completes the proof.

5. A FACTORIZATION LEMMA

In this section a necessary and sufficient condition for the factorization property of n (·) as described in Theorem 2 will be derived. For convenience of notation, a class $\mathcal{K}$ of real-valued functions is introduced.

DEFINITION: Let $\mathcal{K}$ be the class of absolutely continuous real-valued functions k(·) defined on $[0,\infty)$ with each k(·) satisfying $0 < \inf k(t) \leq \sup k(t) < \infty$.

LEMMA 4: Let $k(\cdot) \in \mathcal{K}$ be given and let σ be a non-negative constant. The following two statements are then equivalent:

(i) There is a constant $K > 0$ such that

$$\left| \frac{1}{t} \int_0^t \left| \frac{k'(t)}{k(t)} - 2\sigma \right| dt - 2\sigma \right| \leq \frac{K}{t}$$

for all $t > 0$.

(ii) There are functions $k_+(\cdot)$ and $k_-(\cdot)$ both in K satisfying the following properties for all $t \geq 0$:

(a) $k(t) = k_+(t) \cdot k_-(t)$

(b) $k_+(t) \exp(-2\sigma t)$ is monotone non-increasing

(c) $k_-(t)$ is monotone non-decreasing.

It is somewhat simpler to write out the proof of the following equivalent version of Lemma 4 (obtained by considering log k(·)).

LEMMA 4: Let $\ell(\cdot)$ be a real-valued absolutely continuous bounded (above and below) function of t, for t in $[0,\infty)$. Then the following two statements are equivalent:

(i') There is a constant $K > 0$ such that

$$2\sigma t - K \leq \int_0^t \left| \ell'(t) - 2\sigma \right| dt \leq 2\sigma t + K$$

for all $t \geq 0$.

(ii') There are two real-valued absolutely continuous bounded (above and below) functions $\ell_+(\cdot)$ and $\ell_-(\cdot)$, defined on $[0,\infty)$, with the following properties

(a') $\ell(\cdot) = \ell_+(\cdot) + \ell_-(\cdot)$

(b') $\dfrac{d\ell_+(t)}{dt} \leq 2\sigma$ a. e. in t

(c') $\dfrac{d\ell_-(t)}{dt} \geq 0$ a. e. in t

PROOF OF LEMMA 4': Assume at first that (i') holds:

$$\text{Define } \ell_+(t) \triangleq \ell(0) + \int_0^t \frac{\ell'(t) + \sigma - |\ell'(t) - 2\sigma|}{2} dt$$

$$\triangleq \ell(0) + \int_0^t \operatorname{Min}\left\{\ell'(t),\ 2\sigma\right\} dt \tag{23}$$

and

$$\ell_-(t) \triangleq \int_0^t \frac{\ell'(t) - 2\sigma + |\ell'(t) - 2\sigma|}{2} dt$$

$$\triangleq \int_0^t \operatorname{Max}(\ell'(t) - 2\sigma, 0)\, dt. \tag{24}$$

Then with these definitions it is easy to check that (a'), (b') and (c') hold. What is left to be shown is that $\ell_+(\cdot)$ and $\ell_-(\cdot)$ are bounded functions.

But

$$\ell_+(t) = \ell(t) + \sigma t - \frac{1}{2}\int_0^t \left|\ell'(t) - 2\sigma\right| dt$$

while

$$\ell_-(t) = \ell(t) - \ell(0) + \frac{1}{2}\int_0^t \left|\ell'(t) - 2\sigma\right| dt - \sigma t.$$

Assumption (i') and the fact that $\ell(t)$ is bounded assure that $\ell_+(\cdot)$ and $\ell_-(\cdot)$ are bounded functions and this completes the proof of (i') $\rightarrow$ (ii').

In the other direction assume (ii') holds. Then it follows that

$$\frac{d\ell_-(t)}{dt} \geq \operatorname{Max}(\ell'(t) - 2\sigma, 0) \triangleq \frac{(\ell'(t) - 2\sigma) + \left|\ell'(t) - 2\sigma\right|}{2} \tag{25}$$

for almost all $t \geq 0$.

To see this, note that if (25) did not hold, there would exist a set of positive measure on which

$$0 \leq \frac{d\ell_-(t)}{dt} < \ell'(t) - 2\sigma.$$

But

$$\frac{d\ell(t)}{dt} = \frac{d\ell_+(t)}{dt} + \frac{d\ell_-(t)}{dt} \qquad \text{a. e.}$$

so that one would have

$$0 \leq \frac{d\ell(t)}{dt} - \frac{d\ell_+(t)}{dt} < \frac{d\ell(t)}{dt} - 2\sigma$$

on some set of positive measure. This, of course, would imply that $\frac{d\ell_+(t)}{dt} > 2\sigma$ on a set of positive measure which would then violate (b'). Thus (25) is verified.

Now integrating both sides of (25) from 0 to t yields:

$$\ell_-(t) - \ell_-(o) \geq \frac{\ell(t) - \ell(o)}{2} - \sigma t + \frac{1}{2}\int_o^t \left|\ell'(t) - 2\sigma\right| dt$$

valid for all $t \geq 0$. Noting that both $\ell_-(\cdot)$ and $\ell(\cdot)$ are bounded (above and below) by assumption (ii'), the estimate

$$\int_o^t \left|\ell'(t) - 2\sigma\right| dt \leq 2\sigma t + K$$

follows for some constant $K > 0$ and all $t \geq 0$.

The inequality

$$\int_o^t \left|\ell'(t) - 2\sigma\right| dt \geq 2\sigma t - K$$

follows easily from the fact that

$$\text{Max}\,(\ell'(t) - 2\sigma, 0) \triangleq \frac{(\ell'(t) - 2\sigma) + \left|\ell'(t) - 2\sigma\right|}{2}$$

is non-negative for almost all $t \geq 0$. With these remarks, the proof of (ii') $\rightarrow$ (i') is complete.

COROLLARY 3: A necessary and sufficient condition for assumption 2 of Theorem 2 to hold is that either:

(a) there is a $K > 0$ such that

$$\left| \frac{1}{t} \int_0^t \left| \frac{\dot{n}(t)}{n(t)\left(1 - \frac{n(t)}{k}\right)} - 2\sigma \right| dt - 2\sigma \right| \leq \frac{K}{t} \qquad \text{for all } t > 0$$

or else

(b) there is a $K > 0$ such that

$$\left| \frac{1}{t} \int_0^t \left| \frac{\dot{n}(t)}{n(t)\left(1 - \frac{n(t)}{k}\right)} + 2\sigma \right| dt - 2\sigma \right| \leq \frac{K}{t} \qquad \text{for all } t > 0.$$

PROOF: Apply Lemma 4 to

$$n^*(t) \triangleq n(t)\left(1 - \frac{n(t)}{k}\right)^{-1}$$

to get (a) or apply Lemma 4 to

$$\frac{1}{n^*(t)} \triangleq \left(1 - \frac{n(t)}{k}\right)\frac{1}{n(t)}$$

to get (b).

CONCLUDING REMARKS

The techniques developed here show promise of eliminating the "multiplier" dependence from many recent stability results. In particular, investigations are currently under way on problems involving monotone non-linearities (See [2]) with a view towards obtaining geometric criteria there.

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